ON THE PROBLEM OF THE STABILIZATION OF A MECHANICAL SYSTEM

(K ZADACHE O STABILIZATSII MEKHANICHESKOI SISTEMY)

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This paper deals with the problem of the stabilization of a mechanical system, in the neighborhood of its position of stability, by using some additional forces [1 to 3]. It considers the problem of stabilization and control by signals depending upon the velocity. It relates the problem of the stabilization by dissipative forces [2] to the problem of the analytical designing of an optimum control system [4] and with the characteristics of controllability and predictability of the system [5 and 6] from its specific coordinate [3]. The effect of the dissipative and gyroscopic forces on the controllability, predictability and ability to be stabilized is studied. Formulation of the "maximum-minimum" rule [6 to 8] which determines in the case of the linear approximation the optimum command u from the smallest intensity $\rho^*[u]$.

1. We shall consider a holonomic mechanical system described by Equation

$$\frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = Q_i\,(t, q_1, \ldots, q_n, u) \qquad (i = 1, \ldots, n) \tag{1.1}$$

where q_i are the curvilinear coordinates, T is the kinetic enegy, Q_i are the generalized forces and u is the control signal. The quantity u in Equations (1.1) is considered to be a scalar.

Let system (1.1) have a solution $q_i = 0$ for $u \equiv 0$. We shall assume that the linear approximation of (1.1) is stationary and has the form

$$\sum_{j=1}^{n} a_{ij}q_j'' = \sum_{j=1}^{n} b_{ij}q_j + b_i u \qquad (i = 1, \ldots, n) \tag{1.2}$$

Here, a_{ij}, b_{ij} and b_i are constants, Expression $\Sigma a_{ij}q_iq_j$ is positive definite and $b_{ij} = b_{ji}$. In agreement with the general terminology, we shall say that such a system is conservative.

Without loss of generality, we shall assume that the kinetic energy in the first approximation $\qquad T^0 = {}^1/_2\, \Sigma\, a_{ij}q_i'q_j'$

appears as the sum of the squares of the velocities

$$T^0 = {}^1/_2\,(q_1'^2 + \ldots + q_n'^2)$$

and that only one of the numbers b_i in Equations (1.2) is different from zero, let $b_1 = 1$. This is always possible by a linear transformation of the variables q_i and u. In accordance with this, we shall say that the system is subject to the control of the first coordinate.

We shall designate by lower case latin letters column vectors. The symbol * will designate transformation.

We shall consider the problem of the analytical designing of the control system [4].

P r o b l e m 1.1 . Find a function

$$u = l^*q' = l_1q_1' + l_2q_2' + \ldots + l_nq_n' \tag{1.3}$$

depending only upon the velocities, and such that the solution $q_i = 0$ is asymptotically stable [9], on the basis of equations (1.2) and (1.3), and such that the minimum of the functional

$$J = \int_0^\infty \omega\,(z,\,u)\,dt = \int_0^\infty \Big(\sum_{ij=1}^{2n} c_{ij}z_iz_j + d\,u^2\Big)\,dt = \min \tag{1.4}$$

is obtained for variations of $q_i(t)$, $u(t)$.

Here $d > 0$, $\Sigma c_{ij}z_iz_j$ is a positive semi-definite form $z_{2i-1} = q_i'$, and $z_{2i} = q_i$. We shall transform the system (1.2) into normal coordinates [1]

$$y_i'' = \lambda_iy_i + e_iu \quad (i = 1,\ldots,n) \tag{1.5}$$

Here λ_i and e_i are determined from Equations

$$|B - \lambda E| = 0, \qquad e_i = s_{1i}$$

$$\sum_{k=1}^{n} (b_{jk} - \delta_{jk}\lambda_i)\,s_{ki} = 0 \quad (i,\,j = 1,\ldots,\,n,\ \delta_{ii} = 0,\ \delta_{ij} = 0,\ i \neq j)$$

whereupon the vectors $s_i = \{s_{ki}\}$ $(k = 1,\,\ldots,\,n)$ are the orthogonal eigenvectors of the matrix $B = \|b_{ij}\|$.

We replace Equations (1.5) by the system

$$x'_{2i-1} = \lambda_{2i-1}x_{2i} + s_{1,\,2i-1}u, \qquad x'_{2i} = x_{2i-1} \qquad (i = 1,\ldots,n) \tag{1.6}$$

In coordinates x_i the functional (1.4) becomes

$$J = \int_0^\infty \Big(\sum_{ij=1}^{2n} d_{ij}x_ix_j + du^2\Big)\,dt \tag{1.7}$$

The desired stabilizing signal

$$u = p_1x_1 + p_3x_3 + \ldots + p_{2n-1}\,x_{2n-1} \tag{1.8}$$

and the optimum Liapunov function

$$V = \sum_{ij=1}^{2n} A_{ij}x_ix_j \tag{1.9}$$

which determines that signal, satisfy the equations of Liapunov-Bellman [4 and 10]

$$\sum_{i\,j=1}^{2n} d_{ij}x_ix_j + du^2 + \sum_{k=1}^{n} (\lambda_{2k-1}x_{2k} + s_{12k-1}u)\frac{\partial V}{\partial x_{2k-1}} + \sum_{k=1}^{n} \frac{\partial V}{\partial x_{2k}} x_{2k-1} = 0 \qquad (1.10)$$

$$2du + \sum_{k=1}^{n} s_{12k-1} \frac{\partial V}{\partial x_{2k-1}} = 0$$

Substituting (1.8) and (1.9) into (1.10) and equating to zero the coefficients of identical x_ix_j, we get for the values of p_{2k-1} and A_{ij} the equations

$$\sum_{i=1}^{n} A_{2k,\,2i-1}s_{1,\,2i-1} = 0 \qquad (k = 1,\ldots,\ n) \qquad (1.11)$$

$$\sum_{i=1}^{n} A_{2k-1,\,2i-1}\, s_{1,\,2i-1} + dp_{2k-1} = 0 \qquad (k = 1,\ldots,\ n) \qquad (1.12)$$

$$d_{2k-1,\,2p-1} + dp_{2k-1}\, p_{2p-1} + p_{2k-1}\sum_{i=1}^{n} A_{2i-1,\,2p-1}\, s_{1,\,2i-1} +$$

$$+\ p_{2p-1}\sum_{i=1}^{n} A_{2i-1,\,2k-1}\, s_{1,\,2i-1} + A_{2k-1,\,2p} + A_{2p-1,\,2k} = 0 \qquad (k,\ p = 1,\ldots,\ n) \qquad (1.13)$$

$$d_{2p,\,2k-1} + \lambda_{2p-1}\, A_{2k-1,\,2p-1} + A_{2k,2p} = 0 \qquad (k,\ p = 1,\ldots,\ n) \qquad (1.14)$$

$$d_{2p,2k} + \lambda_{2p-1}\, A_{2k,\,2p-1} + \lambda_{2k-1}\, A_{2p,\,2k-1} = 0 \qquad (k,\ p = 1,\ldots,\ n) \qquad (1.15)$$

We shall consider a class of such problems for which the optimum function V (1.9) can be constructed as the sum of two components, one of which depends only on the coordinates and the other on the velocities only. Then $A_{2k,\,2p-1} = 0$.

We get the equation $V(x^\circ) = J^\circ(x^\circ)$, where $J^\circ(x^\circ)$ is the minimal value of the functional (1.7) considered as a function of the initial conditions of the system $x(0) = x^\circ$. Consequently, we shall consider the class of problems 1.1, for which the minimum value of the functional J° in the function of the initial state of the system, can be broken down into two components, one of which depends only on the initial coordinates $q_{i0} = q_i^{(0)}$ and the other one, only on the initial velocities $q'_{i0} = q'^{(0)}_i$

We shall assume that in the linear approximation [5 and 11] the system (1.1) can be completely controlled by the signal u, and consequently, the following conditions are fulfilled:

$$\lambda_{2i-1} \neq \lambda_{2j-1}, \qquad s_{1,\,2i-1} \neq 0 \qquad (i,\ j = 1,\ldots,\ n;\ i \neq j) \qquad (1.16)$$

With the assumptions made, there follows from Equations (1.11) to (1.15) that the following relations must be fulfilled:

$$\sum_{ij=1}^{2n} d_{ij}x_ix_j + du^2 = 2d\Big(\sum_{k=1}^{n} p_{2k-1}x_{2k-1}\Big)^2, \qquad p_{2k-1} = \mp\frac{\sqrt{d_{2k-1,\,2k-1}}}{d} \qquad (1.17)$$

$$A_{2k,\,2k} = -\,\lambda_{2k-1}\, A_{2k-1,\,2k-1},\ A_{2k-1,\,2k-1} = \pm\frac{\sqrt{dd_{2k-1,\,2k-1}}}{s_{1,\,2k-1}} \qquad \begin{pmatrix} k,\ p = 1,\ldots,\ n \\ k \neq p \end{pmatrix}$$

$$A_{2k,\,2p} = A_{2k-1,\,2p-1} = 0 \qquad (k,\ p = 1, \ldots,\ n;\ k \neq p) \tag{1.18}$$

(the sign in front of the radical in (1.18) is chosen in such a way that $A_{2k-1,\,2k-1} > 0$) .

We shall discuss the sufficient conditions of solvability of the problem 1.1 with our assumptions. The function V determined by (1.9) must be positive definite, i.e. the inequality

$$\lambda_{2k-1} < 0 \qquad (k = 1, \ldots,\ n) \tag{1.19}$$

must be satisfied.

This means that the original system (1.2) must be stable in the absence of control. The fulfillment of conditions (1.19) is sufficient for the functions V to be positive definite.

A positive definite function V , having a negative semi-definite derivative $dV\,/\,dt = -\omega\,(z,\,u)$ will guarantee [12] an assimptotic stability of the motion $x_i = 0$, if, on the surface

$$\frac{dV}{dt} = -2\left(\sum_{k=1}^{n} p_{2k-1}x_{2k-1}\right)^2$$

there are no complete semi-trajectories of the system (1.6),(1.8), except for the equilibrium position $x_j \equiv 0$. A sufficient condition is obtained if the linear forms

$$\frac{d^k}{dt^k}\left(\sum_{i=1}^{n} p_{2i-1}x_{2i-1}\right) \qquad (k = 0,\,1,\,\ldots,\,2n-1)$$

formed on the basis of (1.6),(1.8),are linearly independent. Sufficient conditions for asymptotic stability of the system (1.6),(1.8) can be deduced, as was done in [2]. However, the conditions of Theorem [12] can also be quickly considered from another point of view, by relating this theorem with the property of predictability of the system considered.

On the basis of the system of equations

$$dx\,/\,dt = Ax + bu \qquad (u = p^*x,\ x = \{x_1,\,\ldots,\,x_m\}) \tag{1.20}$$

let some Liapunov function (the quadratic form $V(x)$) have the negative semi-definite derivative

$$dV\,/\,dt = -\mu\,(x) - du^2 \tag{1.21}$$

where $\mu(x)$ is a positive semi-definite form of the variable x_i . The ensemble of points x , for which $\mu(x) = 0$ represents a linear subspace of space $\{x_i\}$. We shall denote this subspace by the symbol N . Together with system (1.20) let us consider the homogenous system

$$dx\,/\,dt = Ax \tag{1.22}$$

for which we shall examine the problem of prediction of the value

$$\eta\,(T) = r^*x\,(T)$$

from the value $\xi\,(t) = u\,(t) = p^*x\,(t)$ for $0 \leqslant t \leqslant T$, i.e. the problem of the operation [13, 6 and 3]

$$\eta\,(T) = \int_0^T \xi\,(\tau)\,\varphi\,(\tau)\,d\tau \tag{1.23}$$

The following statement is true.

L e m m a 1.1 . Let it be possible to choose a positive definite function $V(x)$ having a negative semi-definite derivative (1.21) according to Equation (1.20). If, for the system (1.22), the problem (1.23) of prediction of the quantity $\eta(T) = r^*x(T)$ from the quantity $\xi(t) = u(t) = p^*x(t)$ is possible for all vectors r of the subspace N, then the solution $x = 0$ is asymptotically stable on the basis of Equation (1.20).

In fact, we shall consider a solution $x(t)$, different from $x(t) \equiv 0$ and such that $dV/dt \equiv 0$. According to (1.21), this is possible only if conditions $u[x(t)] \equiv 0$ and $\mu[x(t)] \equiv 0$, are fulfilled, i.e. in any case, only if the solution $x(t)$ lies in the subspace N for all $t \geqslant 0$. As long as $x(t) \not\equiv 0$, we have, at some moment, $t = T$, $x(T) = r \neq 0$ and $r \in N$. Thus $r^*x(T) = r^2 > 0$. From the condition of the Lemma, the value $\eta(T)$ is predictable from the value $u(t)$ $(0 \leqslant t \leqslant T)$. According to [6], this predictability is possible only if $u(t) \not\equiv 0$ when $\eta(T) \neq 0$. The contradiction we have obtained ($u(t) \equiv 0$ and $u(t) \not\equiv 0$) shows that there are no solution $x(t)$, different from $x(t) \equiv 0$, for which we could get $dV/dt \equiv 0$. According to [12], the solution $x = 0$ of the system (1.20) is asymptotically stable, which proves the Lemma.

N o t e 1.1 . The value $\eta = r^*x(T)$ is predictable from the value $\xi = p^*x(t)$ if the vector r^* lies in the subspace W generated by the vectors $p^*, p^*A, \ldots, p^*A^{n-1}$. Consequently a sufficient condition of asymptotic stability of the system (1.20) which has a positive definite function $V(x)$ having a negative semi-definite derivative (1.21), is obtained if the linear subspace N, where $\mu(x) = 0$ is contained in the subspace W.

The asymptotic stability will be guaranteed if the system (1.6) for $u \equiv 0$ is completely predictable from the quantity $\Sigma p_{2k-1}, x_{2k-1}$. A sufficient condition is obtained if conditions (1.16) and (1.19) are satisfied and

$$p_{2k-1} \neq 0 \qquad (k = 1, \ldots, n) \tag{1.24}$$

Thus, the following statement is true.

T h e o r e m 1.1.1. Let conditions (1.16) and (1.19) be satisfied and let the expression $\Sigma d_{ij} x_i x_j$ in the minimized functional (1.7) have the form

$$\sum_{ij=1}^{n} d_{ij}x_i x_j = \Big(\sum_{k=1}^{n} f_{2k-1}\, x_{2k-1}\Big)^2 = (f^*x)^2 \tag{1.25}$$

whereupon all $f_{2k-1} \neq 0$. Then problem 1.1 has the solution $u = p^*x$ (1.8), which stabilizes the system (1.6) in an optimum manner. Thus $p_{2k-1} = d^{-1/2} f_{2k-1}$ and, after the substitution $u(t) = p^*x(t)$, the subintegral function ω in the minimized functional becomes $\omega = 2d(p^*x)^2$.

T h e o r e m 1.1.2. Taking into consideration only the problems where the minimal value of the functional $J^\circ = \min J$, (considered as a function of the initial condition of the system q_{i0}, q'_{i0}), appears as the sum of two components, one of which depends on coordinates q_{i0} only, and the other on the velocities q'_{i0} only, the problem of the stabilization can be solved only if conditions (1.16), (1.19) and (1.25) are fulfilled. Therefore these conditions are also necessary conditions of solvability of the problem.

2. We shall consider the relation between the results we have just obtained and the problem of stabilization of a system with dissipative forces [2].

Thus, let us assume the system (1.5) is stable for $u \equiv 0$, i.e. $\lambda_i < 0$ $(i = 1, \ldots, n)$. Then, according to [3], the equilibrium position $y_i = 0$ can be made asymptotically stable by means of the choice of a force $u(y, y')$ of an arbitrary kind, if, and only if the conditions of controllability (1.16) are fulfilled. The force $u(y, y')$ can be determined with these conditions from the solution of the problem of the analytical designing of an optimum control system [4] by minimizing the value (1.4). We shall choose the value J, in accordance with the results of Section 1, in the form

$$J = \int_0^\infty \omega(x, u)\, dt = \int_0^\infty \left[\left(\sum_{i=1}^n s_{1,2i-1} x_{2i-1}\right)^2 + u^2\right] dt \tag{2.1}$$

The optimum Liapunov function $V(x)$ and the optimum command $u^\circ(x)$, satisfying the equation of Liapunov-Bellman

$$\min_u \left(\frac{dV}{dt} + \omega(x, u)\right) = 0 \tag{2.2}$$

as follows from Section 1, have the form

$$V(x) = \sum_{i=1}^n (x_{2i-1}^2 - \lambda_{2i-1} x_{2i}^2), \qquad u^\circ(x) = -\sum_{i=1}^n s_{1,2i-1} x_{2i-1} \tag{2.3}$$

whereupon, on the basis of Equation (1.6), we have for $u = u^\circ$

$$\frac{dV}{dt} = -2\left(\sum_{i=1}^n s_{1 2i-1} x_{2i-1}\right)^2 \tag{2.4}$$

When conditions (1.16) and (1.19) are fulfilled, the system (1.6) (2.4) satisfies the conditions of Lemma 1.1, i.e. in fact, the Liapunov function $V(x)$ (2.3),(2.4) guarantees an asymptotic stability of equilibrium $x_i = 0$. However the values $s_{1,2i-1} u^\circ$ in Equations (1.5) can be considered as generalized dissipative forces X_i, generated by the function of Rayleigh [14].

$$2R = \left(\sum_{i=1}^n s_{1,2i-1} x_{2i-1}\right)^2 \tag{2.5}$$

In such a case, the optimum Liapunov function $V(x)$ is equal to twice the energy of the system

$$H = \frac{1}{2}\left(\sum_{i=1}^n x_{2i-1}^2 - \sum_{i=1}^n \lambda_{2i-1} x_{2i}^2\right)$$

In coordinates q_i, q_i', described by Equations (1.2), we have

$$2R = (q_1')^2, \qquad Q_1 = u^\circ = -q_1'$$

Consequently, if we assume that the system (1.1) is controlled in the linear approximation by a signal on the coordinate q_1, we shall deduce that when conditions (1.16) and (1.19) are fulfilled, the stabilization of the system (1.2) is possible by a dissipative force $u^\circ = -\partial R/\partial q_1$, generated by a partial dissipation $R(q)$ on the coordinate q_1. (The conditions of controllability (1.16) mean, in particular, that the generalized force Q does not coincide in direction with any canonical axis x_i).

Thus, the following statement is true.

Theorem 2.1. The stable system (1.2), subject to the command u on its first coordinate q_1 can be stabilized to asymptotic stability by a force of an arbitrary kind $u(q, q')$ if, and only if the stabilization of the system is possible by a dissipative force $u(q_1') = -\partial R / \partial q_1'$. Therefore, the conditions of stabilization by a dissipative force [2] coincide with the general conditions of full controllability and ability to be stabilized of the system [3]. Thus, a dissipative force $u(q_1')$ can be considered as the solution of the problem of the analytical designing of an optimum control system u, which minimizes the quantity

$$J = \int_0^\infty [q_1'^2 + u^2(t)]\, dt = \int_0^\infty [2R(q_1') + u^2(t)]\, dt \qquad (2.6)$$

The minimum value J° of quantity J of (2.6) is therefore equal to twice the quantity of energy dissipated during the decaying motion of the optimum system

$$J^\circ = \int_0^\infty 4R\,[q_1'(t)]\, dt$$

The problem 1.1, as studied in the case of the first approximation (1.2) can be extended, on the basis of the general results [15 to 17], to the analogous problem of the completely nonlinear system (where nonlinear elements include the small terms of highest order with respect to t). We shall not consider here the details of this extension.

We shall notice only that a conservative stationary system has an integral of energy, but for the motion of the controlled system

$$\frac{d}{dt}\frac{\partial T}{\partial q_1'} - \frac{\partial T}{\partial q_1} = \frac{\partial \Pi}{\partial q_1} + u, \qquad \frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = \frac{\partial \Pi}{\partial q_i} \qquad (i = 2, \ldots, n) \qquad (2.7)$$

the derivative of the complete energy of system H, with inclusion of its nonlinear elements, satisfies condition

$$dH/dt = q_1' u \quad \text{or} \quad dH/dt = -2R \quad \text{for} \quad u = -\partial R/\partial q_1', \quad 2R = q_1'^2$$

Therefore, the statement of Theorem (2.1) is still valid in the nonlinear case.

Consequently, the solution of the problem of analytical designing of an optimum control system

$$\int_0^\infty [2R(q_1') + u^2(t)]\, dt = \min \qquad (2R = q_1'^2)$$

is also determined in the nonlinear case by the optimum Liapunov function

$$V(q, q') = 2H(q, q')$$

equal to twice the complete energy, whereupon $u^\circ = -\partial R/\partial q_1'$.

3. We shall consider the influence of dissipative forces on the controllability and predictability of conservative mechanical systems in the neighborhood of the equilibrium position. We shall assume that system (1.6) is not controllable completely by the control u. This means that the conditions (1.16) are not fulfilled, i.e. some of the λ_i are equal, or some of the numbers s_{1i} are equal to zero.

We shall consider the particular case in which $\lambda_1 = \ldots = \lambda_n = \lambda \neq 0$. Therefore we shall consider the system

$$x'_{2i-1} = \lambda x_{2i} + \alpha_i u, \qquad x'_{2i} = x_{2i-1} \qquad (i = 1, \ldots, n) \qquad (3.1)$$

where at least one of the numbers $\alpha_i \neq 0$. We shall examine the existence of such dissipative forces when the system (3.1) becomes completely controllable by the control u. In order to find a solution to this problem, it is sufficient to show that we can find a positive semi-definite function

$$2R = -\sum_{ij=1}^{n} \gamma_{ij} x_{2i-1} x_{2j-1} \qquad (3.2)$$

such that the system

$$x'_{2i-1} = \lambda x_{2i} - \frac{\partial R}{\partial x_{2i-1}} + \alpha_i u, \qquad x'_{2i} = x_{2i-1} \qquad (i = 1,\ldots, n) \qquad (3.3)$$

becomes controllable completely by the control u. In the present case, we can determine some dissipative forces making the system (3.1) completely controllable in the following manner. The system (3.1) can always be transformed into the form

$$z'_{2i-1} = \lambda z_{2i} + \beta_i u, \qquad z'_{2i} = z_{2i-1} \qquad (\beta_i \neq 0, \quad i = 1,\ldots, n) \qquad (3.4)$$

Let us choose a function R of the form

$$2R = \sum_{i=1}^{n} \mu_i z^2_{2i-1}$$

The system (3.4) becomes

$$z'_{2i-1} = \lambda z_{2i} - \mu_i z_{2i-1} + \beta_i u, \qquad z'_{2i} = z_{2i-1} \qquad (i = 1,\ldots, n) \qquad (3.5)$$

It can be verified that the system (3.5) is controllable completely if $\mu_i \neq \mu_j$, i.e. it is always possible, in this case, to find some dissipative forces which make the system completely controllable.

4. Example. Let us consider a simple illustrative example. Let a material point move on the surface $z = f(x,y)$. The coordinates (x,y,z) are orthogonal, the z-axis directed vertically upwards and x and y along the principal directions of the surface at the point O, which, we assume, is the extremum point of the function $z = f(x,y)$. We shall suppose that the deviation of the material point from the position of equilibrium (0,0,0) and its velocity are small. We shall assume that the meterial point is subject to the force of gravity and the control u, which has a constant horizontal direction. The equations of motion of the point are in the first order approximation

$$x'' = \lambda_1 x_1 + \alpha_1 u, \qquad y'' = \lambda_2 y + \alpha_2 u \qquad (4.1)$$

a) If $\lambda_1 \neq \lambda_2, \alpha_1 \neq 0, \alpha_2 \neq 0$, the system (4.1) can be completely controlled and stabilized by the control u, if the point O of the surface $z = f(x,y)$ is not a spherical point, and if the control is not directed along the principal directions of the surface at this point;

b) If $\lambda_1 = \lambda_2 = \lambda \neq 0$, i.e. if the point O is a spherical point, the system (4.1) is not controllable completely, and cannot be stabilized for $\lambda > 0$. From Section 3, there follows that, if in such case an irregular friction is applied, the system becomes completely controllable and can be stabilized by the control u if the direction does not coincide with the principal axes of the ellipse of friction. Here, by "ellipse of friction" we mean the curve $R = c^2$, where $R(x',y')$ is the dissipation function.

Let us see how important gyroscopic forces are in problems of control, prediction and stabilization. It is known [1], that a conservative unstable system can be stabilized in many cases (but not asymptotically) by the superposition of adequate gyroscopic forces. This stability is destroyed when dissipative forces are applied [1]. It turns out that gyroscopic forces play, with respect to the controlled system, a role which is as important. In other words, in many cases, an adequate choice of gyroscopic forces will improve the qualities of controllability, predictability and ability to be stabilized, of a conservative mechanical system in the neighborhood of its position of equilibrium.

5. We shall consider the particular case in which we have in (1.6) $\lambda_1 = \ldots = \lambda_n = \lambda$, and at least one of the numbers $a_{1i} \neq 0$. The system (1.6) can then be written in the form

$$x_1' = \lambda x_2 + u, \quad x_2' = x_1, \quad x'_{2i-1} = \lambda x_{2i}, \quad x'_{2i} = x_{2i-1} \quad (i = 2,\ldots, n) \; (5.1)$$

The system (5.1) is not completely controllable by the control u, however it is possible to find gyroscopic forces which would make the system controllable completely.

In fact it can be verified easily that the system

$$x_1' = \lambda x_2 + \omega_1 x_3 + u, \qquad x_2' = x_1 \tag{5.2}$$

$$x'_{2i-1} = \lambda x_{2i} + \omega_i x_{2i+1} - \omega_{i-1} x_{2i-3}, \qquad x'_{2i} = x_{2i-1} \qquad (i = 2, \ldots, n, \ \omega_n = 0)$$

is controllable completely by the control u if

$$\lambda \neq 0, \qquad \omega_i \neq 0 \qquad (i = 1, \ldots, n-1)$$

We shall consider an example in which not all λ are equal. Let us have a pendulum with two degrees of freedom moving in the neighborhood of its upper unstable point of equilibrium, and subject to the horizontal control force lying in the horizontal plane $\{x, y\}$, whereupon we shall suppose that the equations of the first approximation are

$$x'' = \lambda_1 x, \qquad y'' = \lambda_2 y + u \tag{5.3}$$

where x and y are the coordinates of the mass center m rigidly fixed on the shaft which does not rotate around its longitudinal axis.

This system is not completely controlled by the control u along y, and consequently the position $x = y = 0$ cannot be stabilized by any choice of $u\ (x, x', y, y')$. Similarly, this system is not completely predictable from the coordinate y. We shall assume, therefore, that the mass m is concentrated in a flywheel rotating around the longitudinal axis of the shaft with an angular velocity ω. Then, complementary terms determined by the gyroscopic effect appear in Equation (5.1). Let, in such case, the equations of motion of the first approximation take the form

$$x_1' = x_2, \qquad x_2' = \lambda_1 x_1 + \omega x_4, \qquad x_3' = x_4, \qquad x_4' = \lambda_2 x_3 - \omega x_2 + u \tag{5.4}$$

It is easy to check that for $\lambda_1 \neq 0$, $\omega \neq 0$, the system (5.4) is completely controllable by the force u, and consequently, the resulting mechanical system can now be asymptotically stabilized by the force $u\ (x, x', y, y')$. Similarly, the system (5.4) will be predictable (for instance from the quantity $x_3(t)$).

Thus, it can be seen from what has been said previously, that the system which could be neither controlled nor stabilized by a given force, could be controlled and stabilized by the same force when adequate gyroscopic forces are applied.

We shall note, furthermore, an interesting characteristic which distinguishes the property that a conservative system initially stable can be stabilized by a control u, and that a noncontrollable conservative system initially unstable can be controlled and stabilized in presence of gyroscopic forces. This property is closely related to an analogous property which concerns the case of noncontrollable systems. It is shown in Sections 1 and 2 that a conservative system which is stable (but not asymptotically) can be stabilized by a force chosen in a class of forces of an arbitrary kind, if and only if, it is limited to a class of dissipative forces. On the contrary, if, by superposition of gyroscopic forces, we stabilize (but not asymptotically) and make controllable by a control u a system which was initially unstable and not completely controllable by u, this means that we can stabilize it only by choosing a force $u\ (q, q')$ of a sufficiently general kind which cannot be limited to a class of dissipative forces. Those dissipative forces would even destroy the nonasymptotic stability already obtained by means of gyroscopic forces.

6. In conclusion, we shall examine, in the case of a conservative mechanical system (1.2), how the "maximum-minimum" condition is transformed [6 and 7] which determines the control u, which in turn brings the system to the given state $q_i = 0$, $q_i' = 0$ using the smallest possible intensity $\rho^*(u)$.

We shall consider the problem only in the linear approximation. Furthermore, in order to simplify the calculations we shall not consider the problem of bringing the system from the given point q_{1o}, q_{1o}' to the equilibrium position $q_1 = 0$, $q_1' = 0$, but on the contrary the problem of bringing the system from the point $q_1 = 0$, $q_1' = 0$ to the point q_{1o}, q_{1o}'. It is obvious that we can get the solution of one problem from the solution of the other by changing the time t into $-t$.

We shall consider the system described by Equations

$$q_1'' = \sum_{j=1}^{n} b_{1j}q_j + u, \qquad q_i'' = \sum_{j=1}^{n} b_{ij}q_j \qquad (i = 2,\ldots, n) \tag{6.1}$$

The problem is the following.

Problem 6.1. Given the interval of time $0 \leqslant t \leqslant T$ the initial (α) and final (ω) states of the system

$$q_i(0) = q_{i\alpha} = 0, \quad q_i'(0) = q_{i\alpha}' = 0, \qquad q_i(T) = q_{i\omega}, \quad q_i'(T) = q_{i\omega}'$$

find the control $u(t)$ $(0 \leqslant t \leqslant T)$ which brings the system (6.1) from the state α to the state ω and has the smallest possible intensity $\rho^*(u)$.

The form of the function $\rho^*(u)$, which yields an estimate of the intensity of the force u which is used, is supposed given. It is also assumed that the function $\rho^*(u)$ corresponds to the calculation of the norm of the linear operation

$$\varphi = \int_0^T \xi(t)\, u(t)\, dt \tag{6.2}$$

considered on any functional space $\{\xi(t),\ 0 \leqslant t \leqslant T\}$, on which the norm $\rho(\xi)$ is given. Then

$$\rho^*(u) = \sup \left[\int_0^T \xi(\tau)\, u(\tau)\, d\tau \text{ for } \rho(\xi) = 1 \right]$$

We shall assume that the system (6.1) is completely controllable, and consequently that the problem 6.1 can be solved for any end conditions α and ω.

In agreement with [6 and 7], in order to solve the problem 6.1, we must consider the motion $y^\circ(\tau)$ $(0 \leqslant \tau \leqslant T)$ of the system which corresponds to (6.1) and has the smallest possible intensity

$$\min_y \rho(b^* y^\circ) = \alpha \tag{6.3}$$

for the end condition $(q^*(T)\, y(T) = 1)$. The sought optimum command $u^\circ(t)$, which solves problem 6.1 is determined from the condition of maximality of operation (6.2) on $\xi^* = b^* y^\circ$, i.e. from condition

$$\int_0^T \xi^\circ(\tau)\, u^\circ(\tau)\, d\tau = \max_u \left[\int_0^T \xi^\circ(\tau)\, u(\tau)\, d\tau \right] \tag{6.4}$$

with condition $\rho^*(u) = 1/\alpha$. Condition (6.4) is the condition of the maximum principle [18] and condition (6.3) is the condition of the minimum [7] which, according to the conditions of the maximum principle, yields the vector $\psi = y$ which guarantees that the system will come exactly to the given state $q(T) = q_0$. (Here b is a $2n$ column vector $b = \{0, 1, \ldots, 0\}$, q is a $2n$ vector $\{q_1, q_1', \ldots, q_n, q_n'\}$).

Let us change to normal coordinates x_i in Equations (6.1). We shall label them differently than in (1.6). In this section especially we shall consider that the velocity has a greater index than the corresponding coordinate.

Consequently, the symbols x_{2k-1} will designate the coordinates and the symbols x_{2k} the velocities. Then system (6.1) becomes

$$x'_{2k-1} = x_{2k}, \qquad x_{2k}' = \lambda_k x_{2k-1} + p_{2k} u \qquad (k = 1, \ldots, n) \tag{6.5}$$

The end conditions of the problem become $x(0) = x^{(\alpha)} = 0$, $x(T) = x^{(\omega)}$, where the $2n$ dimensional vector $x^{(\omega)}$ is related to the $2n$ dimensional vector $q^{(\omega)}$ by a linear transformation.

The system of equations corresponding to the system (6.5) (for $u = 0$) becomes

$$y'_{2k-1} = -\lambda_k y_{2k}, \qquad y_{2k}' = -y_{2k-1} \qquad (k = 1, \ldots, n) \tag{6.6}$$

After relabelling the variables $y_{2k} = z_{2k-1}$, $y_{2k-1} = z_{2k}$ and after reversing the direction of measurement of time, i.e. after replacing $t = T - \tau$, we get from (6.6) Equations

$$z'_{2k-1} = z_{2k}, \qquad z_{2k}' = \lambda_k z_{2k-1} \qquad (k = 1, \ldots, n) \tag{6.7}$$

which coincide with the principal part of system (6.5). We shall denote by $z^{(\omega)}$ the vector obtained from vector $x^{(\omega)}$ for the labelling of coordinates which relates the vectors y and z; we denote by the symbol l the vector $l = \{p_2, 0, \ldots, p_{2n}, 0\}$ which is deduced from vector $p = \{0, p_2, \ldots, 0, p_{2n}\}$ by an analogous relabelling of coordinates.

We shall find the motion $z(t)$ of system (6.7) satisfying to end condition $([z^{(\omega)}]^* z(0)) = 1$ and such that the signal $\xi(t) = (l^* z(t))$ $(0 \leqslant t \leqslant T)$ has the smallest possible intensity $\rho(\xi) = \min$. Then the sought signal $y(\tau)$, determining the optimum command $u(t)$ according to the conditions of the "maximum-minimum" rule (6.3) and (6.4) is related to the vector $z(t)$ by the relations

$$y_{2k}(\tau) = z_{2k-1}(T - \tau), \qquad y_{2k-1}(\tau) = z_{2k}(T - \tau)$$

Thus, we come to the following conclusion.

T h e o r e m 6.1. The optimum control $u^\circ(t)$ solving the problem (6.1) of the control of mechanical conservative system, is determined by the "maximum - minimum" rule conditions (6.3) and (6.4) where the vector $y(\tau)$ describes the motion of the same system (for $u \equiv 0$) in which the coordinates are replaced by the velocities, the velocities by the coordinates and for which the time is reversed $(\tau = T - t)$.

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